ELLIPTIC CURVES MID TERM EXAM

This exam is of **30 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly.

If you have any questions please call me at **+91 98804 59642** or email me at **rameshsreekantan@gmail.com**.

Please copy the following sentence on the first page of your answer sheet and write your name and signature.

I have not used any unfair or illegal means to answer any of the questions in this exam.

1. Let C be a smooth curve over a field K and D a divisor on C. Without using the Riemann-Roch theorem prove

a. $\mathcal{L}(D)$ is a $\bar{K}$ vector space (4)

b. If $\deg(D) \geq 0$, then

$$\ell(D) = \dim_{\bar{K}} \mathcal{L}(D) \leq \deg(D) + 1$$

(6)

2. Let f be a polynomial of degree d in $K[X]$ where $char(K) = 0$. Let C be the curve given by the affine equation

$$C : Y^2 = f(X)$$

Let $\phi : C \longrightarrow \mathbb{P}^1$ be the function given by

$$\phi((X,Y)) = X$$

Then

a. What is the degree of the map ϕ? (3)

b. What are the ramification points of ϕ and what are their indices? (4)

c. What is the genus of C? (3)

3, Let E/K be an elliptic curve given by Weierstrass equation $F(X_0, X_1, X_2) = 0$. Assume $char(K) \neq 2, 3$. Let $P \in E$ and O the origin.

a. Show that $[3]P = O$ if and only if the tangent line at P intersects E only at P. (3)

b. Show that $[3]P = 0$ if and only if the *Hessian matrix*

$$H = \left(\frac{\partial^2 F}{\partial X_i \partial X_j} \right)_{0 \leq i,j \leq 2}$$

satisfies $\det(H) = 0$. (4)

c. Show that $E[3]$ has 9 points. (3)